(where λ_+^2: $u_+ \rightarrow v_-$ means $s \in u_+$, $\lambda_+^2(s) \in v_-$). Further, we obtain

$$\begin{array}{llll} \lambda_+^2: & L_\pm \rightarrow H_\pm & \lambda_-^2: & L_\pm \rightarrow H_\mp \\ & R_\pm \rightarrow S_\mp & & R_\pm \rightarrow S_\pm \\ & H_\pm \rightarrow H_\mp & & H_\pm \rightarrow L \\ & S_\pm \rightarrow R & & S_\pm \rightarrow S_\mp . \end{array} \quad \text{(B4)}$$

The mapping of other parts of the real s axis can be read off Fig. 18. We have defined $\lambda_\pm^2$ such that $\lambda_+^2 \geq \lambda_-^2$ for real $s<0$ and $4<s<(W-1)^2$, but $\lambda_+^2 \leq \lambda_-^2$ for real $s \geq (W+1)^2$. The curves S and H are in fact defined by the conditions (B4).

After these lengthy preliminaries, the properties of $f(s,W^2|\lambda^2)$ on the nearest sheets can be listed as follows:

$$\begin{array}{lll} \text{(a)} & f_{pp}: & \lambda^2 \text{ singular only if } s \in u_\pm;\ \lambda_+^2 \text{ not singular, } \lambda_-^2 \text{ singular } \in w_\pm^2 \\ \text{(b)} & f_{pq}=f_{qp}: & \text{for } s \in v_\pm,\ w_\pm \text{ both } \lambda_\pm^2 \text{ singular;} \\ & & \text{for } s \in u_\pm,\ \lambda_+^2 \in v_\pm^2 \text{ is singular, } \lambda_-^2 \in w_\pm^2 \text{ not singular.} \end{array} \quad \text{(B5)}$$

We remark that for f_{qp}, as s crosses λ_+ from v_+ to u_+, λ_-^2 crosses the λ^2 cut from below between 0 and 4 [cf. (B4)], having been singular on the p sheet in λ^2 for $s \in v_+$. This singularity passes smoothly on the Riemann surface to the q sheet in λ^2 as s enters u_+.

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Logarithmic Singularities in Processes with Two Final-State Interactions*

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The effects of logarithmic singularities in rescattering processes are investigated. The reaction $\pi N \rightarrow \pi\pi N$ is considered, but treated purely as an S-wave, spinless model. A particular triangle graph is analyzed in detail; it contains as an intermediate state the (3,3) nucleon isobar I, which is described as a spinless particle of complex mass. The graph is calculated from a dispersion relation as a function of the mass s of the two pions in the final state, for low values of the over-all c.m. system energy W. The relation is then analytically continued in W. For a narrow range in W, an enhancement of the square of the amplitude is found near $s=4$ (the pion mass is unity). The analogous enhancement also appears in the W channel near $W=I+1$, for a small range of s only, near $s=4$. The prominence of the effect depends on the width of I, being closely connected with the nearness to the physical region of one of the two logarithmic singularities (anomalous thresholds) of the graph: this distance increases sharply with the isobar width. The positions of the singularities are interpreted as the phase-space limits for the simultaneous production of states with mass s and I. The conclusion is that such a "double excitation" process leads to an enhancement of the triangle amplitude only if, in general, s and I fall in certain narrow ranges. The implications of this result for models of the higher resonances in the elastic channel ($\pi N \rightarrow \pi N$) is briefly discussed.

I. INTRODUCTION

UNTIL the rather recent introduction of self-consistent (bootstrap) methods using the N/D formalism,[1] it is fair to say that most calculations of dynamical effects in strong interactions have been single-particle exchange calculations. However, it is worth asking how we may go further, and include rescattering terms, which arise from the fact that in a multiparticle final state more than just one pair of particles may interact strongly. A typical reaction is shown in Fig. 1, in which a pion is produced in pion-nucleon scattering. In the final state $\pi\pi N$, there is the possibility of three interactions: the two πN ones, and the $\pi\pi$. Figure 2 shows a rescattering term representing the production of a pion and a (3,3) nucleon isobar, the isobar then decaying and its decay pion rescattering from the pion. We call the amplitude for this process F. The problem is to calculate F as a function either of the incoming energy W or of the mass of the two pions $\sqrt{s}$.[2]

Graphs similar to Fig. 2 have been discussed quite extensively.[3] Whereas single-particle exchange graphs lead to poles, these give logarithmic singularities—often called anomalous thresholds—in W or s, and some effort has gone into seeing if these singularities lead to observ-

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[1] See, for example, F. Zachariasen, Phys. Rev. Letters **7**, 112, 268 (1961); G. F. Chew, Phys. Rev. **129**, 2363 (1963); and L. A. P. Balazs, *ibid.* **128**, 1939 (1962).

[2] I am indebted to Dr. S. F. Tuan for stimulating my interest in this type of graph. I have been informed by Dr. Tuan that a calculation, similar to that reported here, has been done by Dr. T. T. Wu and himself.

[3] For example, by V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1221 (1961) [English transl.: Soviet Phys.—JETP **14**, 871 (1962)], for τ decay, and by V. V. Anisovich, A. A. Ansel'm, and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **42**, 224 (1962) [English transl.: Soviet Phys.—JETP **15**, 159 (1962)], for pion production reactions near threshold.

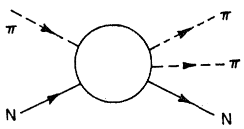


FIG. 1. The process $\pi N \rightarrow \pi\pi N$.

able effects. The first to examine this question were Landshoff and Treiman,[4] but the processes they considered, which involved exclusively stable particles, all had, for one reason or another, very small cross sections. This was remedied by Aaron[5] who showed that if an unstable particle were introduced as an internal line of the graph, which was then calculated as a function of s, there was the possibility of an observable effect at the high end of the s range. Aaron included the unstable particle in the s channel, but, as we shall see below, it is questionable whether a simple application of dispersion theory or perturbation theory is quite correct in that case. We shall include the unstable particle in the W channel, the crossed channel with respect to s, and shall argue that the process can then be calculated straightforwardly, as a function of s.[6]

The reaction represented by Fig. 2 depends on there being two simultaneous strong final-state interactions. Several authors have studied these reactions recently,[7–10] particularly in order to see whether processes such as Fig. 2 lead to an enhancement of one or both of the production ($\pi N \rightarrow \pi\pi N$) and elastic ($\pi N \rightarrow \pi N$) amplitudes. For the one graph we consider, this question can easily be related to the mechanism suggested by Peierls[11] for the generation of the higher elastic pion-nucleon resonances. The structure in the graph, in the W channel, is the one-pion exchange pole in the reaction $\pi+I \rightarrow \pi+\pi+N$, and insofar as this pole leads to singularities apparently near the physically region, one expects some enhancement of F, with a consequent effect in the coupled elastic amplitude. In fact though, our calculation shows that there is no enhancement of F in general, but only for a narrow range of s near the threshold $s=4$ (the pion mass$=1$).

The fact that at least this particular enhancement appears only for a restricted range of the variables near certain thresholds may be relevant to other mechanisms for the generation of elastic resonances. Nauenberg and Pais[12] have suggested that an inelastic threshold itself may produce a "cusp" effect in the elastic channel, while Ball and Frazer[13] have argued that the elastic amplitude will be peaked near a point where the production cross section is rising sharply. From the mechanism we consider, Fig. 2, such a rise in the production cross section, it it occurs at all, does so only near definite thresholds; hence it may serve to accentuate the cusp effect.

We shall calculate F from a dispersion relation in s, the weight function f being evaluated from Cutkosky's rules[14] (Sec. IV). For these manipulations, it is assumed that the isobar may be treated as a particle of complex mass. We find that f has logarithmic singularities at two points s_a and s_b, which are, in general, complex, and whose positions depend on W. These in turn produce singularities of F at s_a and s_b, which for small W are not on the physical sheet of F although one of them s_b may be near the physical region. As W increases, s_a crosses the contour of integration in the dispersion relation, necessitating a deformation of the contour; s_a then appears on the physical sheet of F, although too far from the physical region to produce any effect. In Sec. III we first study the notion of these singularities, deduced equivalently from perturbation theory, in order to understand how to define the dispersion integral in all cases. At the outset, s_a and s_b are introduced in Sec. II by a suggestive kinematical calculation. The results of a numerical evaluation of the integrals is given in Sec. V. Finally, in Sec. VI we summarize the main features of the results, and relate them to the possibility of effects in elastic processes. We emphasize that, throughout, the essential complications of spin and isotopic spin are ignored.

II. THE PHYSICAL NATURE OF THE SINGULARITIES

Cutkosky[14] has given a kinematical argument which makes it plausible that there should be anomalous as well as normal thresholds. Consider the diagram shown in Fig. 2. This represents pion-nucleon scattering with single pion production, via an intermediate state of a pion (mass unity) and a (3,3) nucleon resonance (mass I). We are interested in the behavior of this process, as a function of the mass of the two pions in the final state (s), for a range of values of the initial energy in the over-all c.m. system (W). For the moment, let us ignore the fact that the isobar has a width; we take I to be real. Once W is greater than $I+1$, the diagram may

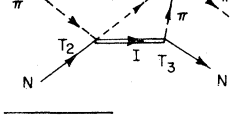


FIG. 2. A rescattering, or triangle graph, contribution to $\pi N \rightarrow \pi\pi N$. The double solid line is a (3,3) nucleon isobar I. T_1, T_2, T_3 stand for the vertices $\pi\pi \rightarrow \pi\pi$, $\pi N \rightarrow \pi I$, $\pi I \rightarrow N$, respectively.

[4] P. V. Landshoff and S. B. Treiman, Phys. Rev. **127**, 649 (1962), hereafter referred to as LT. See also P. V. Landshoff, Phys. Letters **3**, 116 (1962).
[5] R. Aaron, Phys. Rev. Letters **10**, 32 (1963).
[6] F. R. Halpern and H. L. Watson, Phys. Rev. **131**, 2674 (1963), have considered somewhat analogous graphs, from the viewpoint of detection of anomalous thresholds.
[7] C. Bouchiat and G. Flamand, Nuovo Cimento **23**, 13 (1962).
[8] R. F. Peierls and J. Tarski, Phys. Rev. **129**, 981 (1963).
[9] B. d'Espagnat and F. M. Renard, Nuovo Cimento **30**, 556 (1963).
[10] P. K. Srivastava, Phys. Rev. **131**, 461 (1963).
[11] R. F. Peierls, Phys. Rev. Letters **6**, 641 (1961); and S. F. Tuan, Phys. Rev. **123**, 1761 (1962), for application to hyperon resonances.
[12] M. Nauenberg and A. Pais, Phys. Rev. **126**, 360 (1962).
[13] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961).
[14] R. E. Cutkosky, J. Math. Phys. **1**, 49 (1960). We remark that we are using "anomalous" in a general sense to refer to any threshold other than a normal one.

represent a real process—namely, production and subsequent decay of the isobar, with rescattering of its decay pion. There will, however, only be some range of s for which this process is kinematically allowed; and the end points of this range are then, in some sense, thresholds.

The range of s is easy to find. In the c.m. system of the two pions, write the four-vector of the initial state as $(E_1=(W^2+p^2)^{1/2}, \mathbf{p})$ and that of the nucleon (mass M) in the final state as $(E_2=(M^2+p^2)^{1/2}, -\mathbf{p})$, where $p=|\mathbf{p}|$ is the magnitude of the 3-momentum of the nucleon. Then

$$p^2=[s^2-2s(W^2+M^2)+(W^2-M^2)^2]/4s$$

and

$$s=4q^2+4,$$

where q is the magnitude of the 3-momentum $\mathbf{q}$ of the pions. Also,

$$E_1=(s+W^2-M^2)/2\sqrt{s}, \quad -E_2=(s+M^2-W^2)/2\sqrt{s}.$$

Energy conservation at the vertex T_3 gives

$$(E_2+\tfrac{1}{2}\sqrt{s})^2=I^2+p^2+q^2-2pq\cos\theta,$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$. For real θ, s must therefore lie in the range defined by

$$-2pq\leq M^2+1+E_2\sqrt{s}-I^2\leq 2pq. \quad (1)$$

Introducing the variables x, y, and z by

$$s=2(1-x), \quad M^2=I^2+1-2Iz, \quad W^2=I^2+1-2Iy \quad (2)$$

the end points of the range are given by the roots x_a, x_b of

$$x^2+y^2+z^2-2xyz-1=0. \quad (3)$$

Hence

$$x_b=[yz-\{(y^2-1)(z^2-1)\}^{1/2}]\leq x\leq x_a =[yz+\{(y^2-1)(z^2-1)\}^{1/2}] \quad (4)$$

with a corresponding range for s: $s_b\leq s\leq s_a$. These are the usual anomalous thresholds.

Since I is unstable, we have $z>1$, so that for x_a and x_b to be real we need $y<-1$, or $W>I+1$. In other words, when the incident energy is such that the pion-isobar state can be formed, there is a range of s for which 3-momentum conservation also may be satisfied, so that both the $(2\pi,N)$ and the (π,I) states propagate freely. It is now simple to interpret this range on a Dalitz plot for the three-particle state $\pi\pi N$, taking the two mass variables to be those associated with the πN and the $\pi\pi$ systems, $\sqrt{t}$ and $\sqrt{s}$ respectively, say. For a given W, the plot extends over a limited region in the $\sqrt{s}-\sqrt{t}$ plane, which is just that corresponding to Eq. (1). The range of s that we have found is the one for which, given a value of W, the line $\sqrt{s}=$constant, may interest the isobar band $\sqrt{t}=I$. This is illustrated in Fig. 3. R, the boundary of the elliptical-shaped region, is a function of W, and hence so are s_a and s_b.

We may refer to the end points of this range as

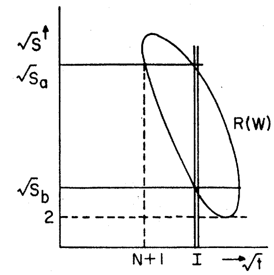


FIG. 3. Dalitz plot in the $\sqrt{s}$, $\sqrt{t}$ plane. The double line is the isobar band $\sqrt{t}=I$.

"double excitation," rather than anomalous, thresholds. At one or both of them, we may expect some type of threshold singularity. Perturbation theory implies (see Sec. III) that the singularities are logarithmic, while the dispersion relation calculation of Sec. IV shows that, to avoid an infinity in the physical region, we have to give I an imaginary part, representing the width of the isobar. Our purpose is to see what effect the resulting pattern of singularities has on the amplitude. First, we discuss their motion in detail.

III. THE SINGULARITIES IN PERTURBATION THEORY

A. Treatment of the Isobar

We now regard Fig. 2 as a perturbation theory graph, assuming that the isobar may be treated as a particle of complex mass, the finite (negative) imaginary part representing the width.

This procedure may seem questionable. One difficulty is the following. One might say that corresponding to the two-particle pion-isobar state, there should be a branch point in W^2 at $(I+1)^2$, on the first (physical) sheet of the two-sheeted surface defined by the elastic cut at $(M+1)^2$. However, it is known[15] that this branch point is not on the physical W^2 sheet, but rather on the sheet reached by crossing the real W^2 axis between the 2- and 3-pion production thresholds. Hence, a simple replacement of the isobar by a complex mass may be misleading. In our case, however, the isobar occurs in a crossed channel with respect to the variable of interest s, so that we might hope that this type of objection would not arise.

In fact, an analysis can be made in terms of the square of the internal mass λ^2 of the pion-nucleon system forming the isobar.[16] The full amplitude, including states of all values of λ^2, is written as

$$g(s,W^2)=\int_c d\lambda^2\sigma(\lambda^2)F(s,\lambda^2,W^2), \quad (5)$$

where $\sigma(\lambda^2)$, essentially the spectral function of the pion-nucleon propagator, has a square-root cut from $(M+1)^2$ to $+\infty$, and where c is a contour along the real

[15] D. Zwanziger, Phys. Rev. **131**, 188 (1963). See also Ref. 16.
[16] I. J. R. Aitchison and C. Kacser, Phys. Rev. **133**, B1239 (1964), preceding paper.

axis above this cut. F is the amplitude for the case in which the square of the mass of the πN system is λ^2. The isobar is included as a pole in $\sigma(\lambda^2)$, on the second λ^2 sheet of the propagator, below the real axis at $\lambda^2=I^2$. The meaning then ascribed to Fig. 2 is that it represents $F(s,I^2,W^2)$, the residue of $F(s,\lambda^2,W^2)$ at the pole $\lambda^2=I^2$, and the continuation to complex values of λ^2 of the conventional amplitude with λ^2 real. Thus g contains F as a part.

The result of this analysis is that peaking effects, due to singularities of g near the physical region, are correctly calculable by treating the isobar as a particle of complex mass—that is, the *nearby* singularities of $F(s,I^2,W^2)$ and $g(s,W^2)$ are the same. On the other hand, this may be by no neans true of the *distant* singularities—although of course, by definition, these give no physical effect, apart from providing a smooth background.

In the following we treat $F(s,W^2)$ exclusively (suppressing from now on the I^2 dependence). We must remember, however, that it is finally only the peaking effects which are to be taken as representing correctly the behavior of g; the remaining background we are unable to calculate properly.

B. The Motion of the Singularities

The locations of the singularities of the graph of Fig. 2 in perturbation theory are well known.[17] They lie on the surface Σ defined by

$$x^2+y^2+z^2-2xyz-1=0$$

exactly the equation [Eq. (3)] defining the points x_a, x_b, of Sec. II. In the present application, z is a fixed complex number, and y is a complex linear function of W^2. [cf. Eqs. (2).] The singularities s_a, s_b for various W^2 are then given by the roots $x(y)$ of Σ for a given z. We emphasize that s_a and s_b are functions of W^2, though we shall usually not indicate this explicitly.

We review the results of Ref. 17. Let $x=x_1+ix_2$, $y=y_1+iy_2$, $z=z_1+iz_2$. The surface Σ is a 4-variety in the 6-dimensional (xyz) space defined by

$$x_1^2+y_1^2+z_1^2=2x_1y_1z_1-1$$
$$-(x_2^2+y_2^2+z_2^2-2x_1y_2z_2-2x_2y_1z_2-2x_2y_2z_1)=0, \quad (6)$$
$$x_1x_2+y_1y_2+z_1z_2-x_2y_1z_1-x_1y_2z_1-x_1y_1z_2=0. \quad (7)$$

Consider first the mapping of the real y axis in Σ. This is a quartic in the x plane, for fixed z, with four double points $\pm z$, $\pm z^*$. It therefore degenerates[18] into two conics, an ellipse and a hyperbola, which we may write as

$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1, \quad \text{and} \quad \frac{x_1^2}{c^2}+\frac{x_2^2}{d^2}=1$$

[17] A detailed account with full references is given by G. Bonnevay, I. J. R. Aitchison, and J. S. Dowker, Nuovo Cimento **21**, 1001 (1961). Note that in this paper a complex value for x, y and z was associated with a complex external, rather than internal, mass.

[18] George Salmon, *Higher Plane Curves* (Chelsea Publishing Company, New York, 1879), 3rd ed., p. 29 ff.

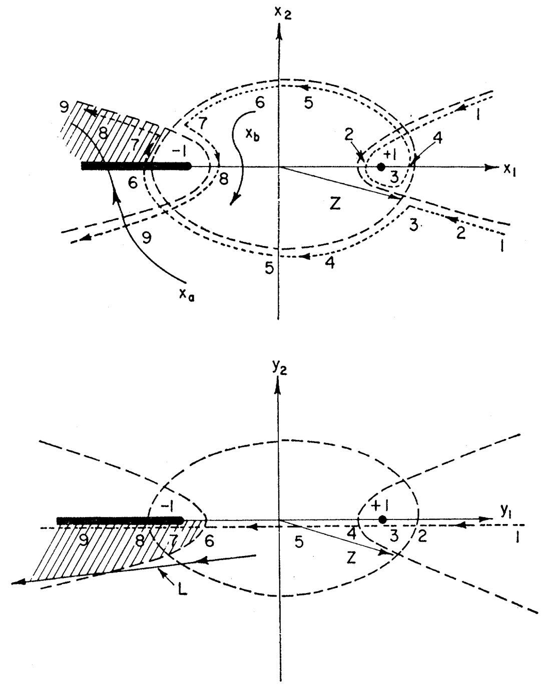


FIG. 4. The mapping Σ in the complex x and y planes. The degenerate quartics forming the mappings of the real axes are shown by long dashed lines. The mapping, in the x plane, of a line in the y plane just below the axis is shown by short dotted lines, corresponding points being indicated by numbers. The mapping in the x plane of the line L in the y plane is shown by solid lines labeled x_a and x_b. When a pair (x,y) is in the shaded region, that point is singular on the physical sheet.

where we find

$$a^2=\tfrac{1}{2}[(z_1^2+z_2^2+1)+\{(z_1^2+z_2^2+1)^2-4z_1^2\}^{1/2}],$$
$$b^2=\tfrac{1}{2}[(z_1^2+z_2^2-1)+\{(z_1^2+z_2^2-1)^2+4z_2^2\}^{1/2}],$$
$$c^2=\tfrac{1}{2}[(z_1^2+z_2^2+1)-\{(z_1^2+z_2^2+1)^2-4z_1^2\}^{1/2}],$$
$$d^2=\tfrac{1}{2}[(z_1^2+z_2^2-1)-\{(z_1^2+z_2^2-1)^2+4z_2^2\}^{1/2}].$$

Since Σ is symmetric in x and y, the same curve, in the y plane, shows the mapping of the real x axis in Σ. To resolve ambiguities at the turning points, we give y a small negative imaginary part; the correspondence between the two x roots of Σ and the values of y is shown in Fig. 4 by numbers along the dotted lines. In summary, whenever y lies on the quartic, one of the corresponding x roots is real.

In our case, as W^2 varies, y follows a line such as L, so that x_a and x_b move as shown by the solid lines, appropriately labeled. Now y passes through the ellipse to the left of $y_1=0$ before the hyperbola, so that x_b crosses the real axis before x_a; also x_a crosses at a more negative value than previously.

C. The Two Sheets of the Amplitude

The amplitude of Fig. 2 has a normal threshold (square root) singularity at $s=4$, or $x=-1$; the s plane is cut along the positive real axis from 4 to ∞, defining

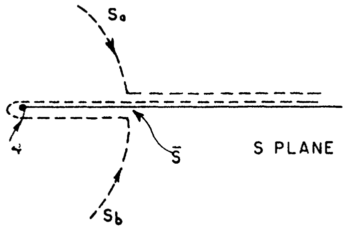


FIG. 5. The motion of the singularities s_a, s_b in the s plane for the case of a real mass of the isobar. s_a and s_b meet at $\bar{s}=[(I+1)^2-M^2]/I$ when $W=I+1$; s_b encircles $s=4$ when $W=2(I^2+1)-M^2$. The arrows show the direction of W increasing.

a two-sheeted surface. Similarly, there are cuts $-1\geq x\geq -\infty$, $-1\geq y\geq -\infty$, shown as thick lines in Fig. 4. The first, physical, sheet in x is defined by $-\pi\leq\arg(x+1)\leq\pi$, and the physical values of $x(s)$ are obtained by approaching the cut from below (above). When we cross a cut, we pass onto a different sheet.

Reference 17 established that a singularity x_a is only on the physical sheet when y is in the lower left quadrant of the hyperbola; this region, and the corresponding singular x region are shown hatched in Fig. 4. As W^2 increases, therefore x_a will appear on the physical sheet, as it crosses the real axis and enters the shaded region. For that W^2 for which x_a lies on the negative real axis, at x_0 say, there will be an infinity in the amplitude at $s_0=2(1-x_0)$. If I, and hence y and z, are purely real, x_0 occurs first at the double point $-z$, for $y=-1$, the hyperbola having collapsed onto the real axis, and x_a and x_b being both on the real axis, and coincident. Then $s_0=\bar{s}=[(I+1)^2-M^2]/I$, for $W=I+1$. Figure 5 shows the motion of the singularities in the s plane for the case I real. This situation has been discussed in rather different contexts by Liu and by Bronzan and Kacser.[19]

The extent of the physical phase space for s is $4\leq s\leq s_1 =(W-M)^2$. Hence, s_0 falls in the physical region if $(I+1)(I-(M+1))(I-(M-1))\geq 0$. If, therefore, I is unstable, there will be an infinity in the amplitude at s_0 when W reaches $I+1$. This difficulty is removed by the procedure of giving I a finite imaginary part I_2. It turns out that s_0 increases rapidly as I_2 becomes different from zero, but is thereafter insensitive to the precise value of I_2. In Fig. 6, the solid lines are paths of s_a and s_b for the (3,3) isobar case: $I=8.91-0.32i$, in units of the pion mass. In these units, $\bar{s}\approx 6$. s_a and s_b now become separated with respect to $\bar{s}$: s_b never reaches the real axis and hardly moves to the right of $s=4$, while s_a crosses far to the right of $\bar{s}$, at $s_0=24.1$ when $W=11.62$. For this W, the singularity s_a is then not in the physical region, which only extends to $s_1=23.62$. This feature is independent of the precise value of I_2; in Fig. 6 we have also plotted the trajectories for $I=8.91-0.02i$ (this represents a width of about 7 MeV for the isobar). Now, s_b follows more closely its path for $I_2=0$, but never reaches the real axis for $s\gtrsim 4$; and s_a, although much closer to the real axis, still crosses outside the physical region.

Perturbation theory, therefore, shows that there is a singularity s_a on the physical sheet, with $\mathrm{Im}s_a<0$, for all W greater than some critical value which depends on I_2 but which is roughly $I_1+1\,(I=I_1+iI_2)$. However, this point is not near the *physical* region of s, which is the limit onto the *top* edge of the cut. It is not expected to produce any effect: The distance between s_a and the physical region has to be reckoned by going around $s=4$, not through the cut. But there remains the singularity s_b; and if I_2 is small, there will be a range of W—corresponding to the range $I+1\leq W\leq[2(I^2+1)-M^2]^{1/2}$ in the case $I_2=0$—for which s_b is singular on the *un*physical sheet *below* the cut; this point is near the physical region. Here again we need a finite I_2 to save us from an infinity of the amplitude; in this case, though, we shall find (in Sec. V) a peaking of the amplitude in the vicinity of s_b for those W's for which s_b is just below the real axis. We remark that if I were stable, $I<M+1$, there would be no W for which either s_a or s_b was near the physical region; this is our reason for studying a graph with an unstable intermediate particle.

IV. CALCULATION OF THE AMPLITUDE FROM A DISPERSION RELATION

A. The Weight Function

The amplitude $F(s,W^2)$ for Fig. 2 is calculated from the dispersion relation

$$F(s,W^2)=\frac{1}{\pi}\int_4^\infty \frac{f(s',W^2)}{s'-s-i\epsilon}ds' \quad . \tag{8}$$

The integration is along a line C just above the real axis. The weight function f is found from Cutkosky's rules,[14] and the result is familiar. We wish, however, to

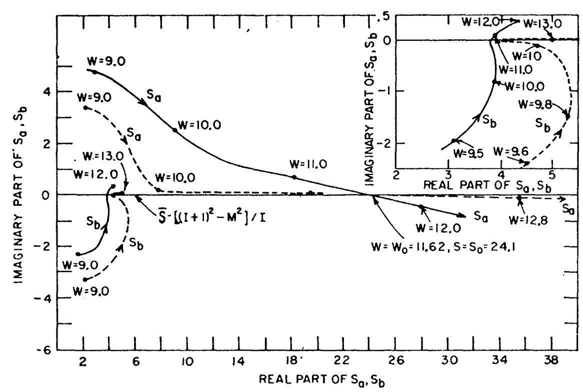


FIG. 6. The motion of s_a and s_b when the isobar is given a complex mass: $I=8.91-0.32i$ (solid line), $I=8.91-0.02i$ (dashed line). s_a crosses the real axis at s_0. The insert shows the region near $s=4$ in more detail.

[19] L. S. Liu, Phys. Rev. **125**, 761 (1962); J. B. Bronzan and C. Kacser, *ibid.* **132**, 2703 (1963); also C. Kacser, *ibid.* **132**, 2712 (1963).

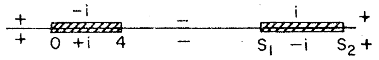


FIG. 7. The cuts of R.

give some of the details, in order to relate this section to Sec. II.

f is proportional to the Feynman amplitude with the two internal pions on their mass shells. In the c.m. system of the two pions, and with the notation of Sec. II,

$$f(s)=\int_{-1}^{1}\frac{q}{4\sqrt{s}}\frac{I}{[I^2-(P+Q)^2]}d\cos\theta d\phi\,, \qquad (9)$$

where P and Q are, respectively, the 4-momenta of the final-state nucleon and the pion associated with the isobar.[20] We have suppressed three factors representing the vertices: We are taking these to be constant.

The denominator in Eq. (9) is

$$D=M^2+1+E\sqrt{s}+2pqz-I^2\,,$$

so that there will be singularities of f when D vanishes at $z=\pm1$, that is, when

$$M^2+1-I^2+E_2\sqrt{s}=\pm 2pq\,.$$

These are just the end points s_a, s_b of the kinematical region described in Sec. II. [cf. Eq. (1).] Nor is this unexpected: f is calculated by requiring the two pions to have their real mass, and the zeros of D correspond to putting the third internal particle I on its mass shell. At the singular points, all the internal particles propagate freely, and these points are just what the calculation of Sec. II gave us. The singularities are logarithmic since the result of the integration in Eq. (9) is

$$f=\frac{\pi}{2(-K)^{1/2}}\ln\frac{a+R}{a-R}\,, \qquad (10)$$

where, in the notation of LT,

$$R=(-KL)^{1/2},\quad -K=(s-s_1)(s-s_2)\,,\quad L=s(s-4)$$
$$a=s^2+s[2I^2-M^2-W^2-2],$$
$$s_1=(W-M)^2,\quad s_2=(W+M)^2.$$

The cuts of f must now be specified. First, R has branch points at 0, 4, s_1 and s_2: the definition of R is shown in Fig. 7. To make explicit the singularities s_a, s_b we rewrite f as

$$f=-\frac{\pi}{(-K)^{1/2}}\ln(a-R)+\frac{\pi}{2(-K)^{1/2}}\ln(a^2-R^2)$$

and we observe that

$$a^2+KL=16sI^2(x^2+y^2+z^2-2xyz-1)\,.$$

[20] This is, of course, the s-wave projection of the isobar pole in the crossed channel of the reaction $\pi\pi\to\pi N\bar{N}$, and the simple explicit appearance of I in the denominator of Eq. (9) is essentially the reason for our being able to calculate straightforwardly with a complex value of I. See Ref. 16.

Hence

$$f=-\frac{\pi}{(-K)^{1/2}}\ln(a-R)$$
$$+\frac{\pi}{2(-K)^{1/2}}\ln 4sI^2(s-s_a)(s-s_b) \qquad (11)$$

and $a-R$ is regular at $s=s_a$. The resulting cuts of f are shown in Fig. 8 for a typical W, below the point at which s_a crosses the real axis: We define the logarithms to be on their principal branches when W is below $\mathrm{Re}(I)+1$. We refer to this as the normal case.

B. The Dispersion Relation in the Normal Case

Consider now a W less than $\mathrm{Re}(I)+1$ for which, referring to Fig. 6, s_a and s_b have finite imaginary parts of opposite signs. For this case, the result of perturbation theory is that F is not singular on its physical sheet, so that the integral in Eq. (8) may consistently be taken along the real axis: s_a and s_b, though singularities of f, are not so of F. This may also be verified for such values of W by making successive continuations in y and z, starting from a real stable value for I, for which the statement certainly holds.

However, as W increases to W_0 say, we notice that, given the determination of R by Fig. 7, $(a+R)$ may vanish at some point s_0 in the range (s_1,s_2), since a has a negative imaginary part. Hence f will be undefined at s_0. In fact, s_0 is the point at which s_a crosses the real axis, and it is, as we saw in Sec. III, indeed in the range (s_1,s_2). We next consider how to modify the representation of Eq. (8) to include such higher values of W.

C. Continuation of the Representation in W

We follow the method of Mandelstam,[21] and continue the representation (8) in the energy W. Referring to Fig. 6 and to Eq. (11) for f, we see that as W increases, the root s_b encircles $s=4$ from below, never crossing the integration contour, so that $\arg(s-s_b)$ goes smoothly from positive to negative values. On the other hand, s_a moves towards the integration path at s_0, to the right of s_1. To continue Eq. (8), we have to distort the contour C downwards into the lower half s plane (into the second s sheet) away from the advancing s_a, as shown in Fig. 9. Furthermore, as s_a crosses, $\arg(s-s_a)$ changes from $-\pi$ to π for all points to the left of s_0, but is continuous to the right of s_0. To make the correct continuation, we note that the cut L_a, attached to s_a,

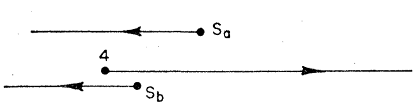


FIG. 8. The cuts of f in the case $\mathrm{Im}(s_a)>0$.

[21] S. Mandelstam, Phys. Rev. Letters 4, 84 (1960).

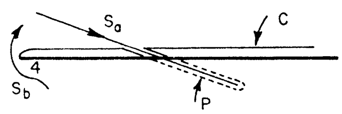


FIG. 9. The deformation C down into the second s sheet (shown by the dotted part of C) to avoid the advancing singularity s_a.

has swept over the integration contour to the left of s_0, so that those points are now on the second sheet of L_a; hence we have to subtract $2\pi i$ from the value of $\ln(s-s_a)$. We can, of course, equally well define L_a as shown in Fig. 10: a piece between s_0 and s_a in the lower half plane, following the trajectory of s_a, and the rest

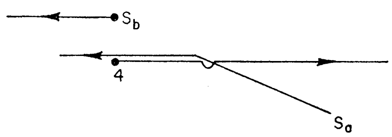


FIG. 10. The cuts of f in the case $\operatorname{Im}(s_a)<0$.

along the real axis $-\infty \leq s \leq s_0$. Then, for $s \leq s_0$, we do not have to subtract the $2\pi i$. For the machine calculation described below, the first definition of L_a is used consistently.

The representation (8) then becomes,[22] using Eq. (11), and the definition of R,

$$F(s,W^2)=\frac{1}{\pi}\int_4^{s_1}\left(f(s')+\frac{2\pi i}{(-K(s'))^{1/2}}\right)\frac{ds'}{s'-s-i\epsilon}+\frac{1}{\pi}\int_{s_1}^{s_0}\left(f(s')-\frac{2\pi}{(+K(s'))^{1/2}}\right)\frac{ds'}{s'-s}+\frac{1}{\pi}\int_{s_0}^{\infty}\frac{f(s')}{s'-s}ds'+F_A, \quad (12)$$

where

$$F_A=\frac{1}{\pi}\oint_P\left(f(s')-\frac{2\pi}{(K(s'))^{1/2}}\right)\frac{ds'}{s'-s}$$

and P is the dotted path of Fig. 9. Since $f(s')$ increases by $2\pi/(K(s'))^{1/2}$ on encircling s_a, we have

$$F_A=\frac{1}{\pi}\int_{s_0}^{s_a}\frac{-2\pi}{(+K(s'))^{1/2}}\frac{ds'}{s'-s}.$$

The integrals not involving f can be done exactly, and one finds

$$F(s,W^2)=\frac{1}{\pi}\int_4^{s_1}\frac{f(s')}{s'-s-i\epsilon}ds'+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{f(s')}{s'-s}$$
$$+\frac{2i}{((s_1-s)(s_2-s))^{1/2}}\ln\left\{\left(1+i\left[\frac{(s_2-s)(s_a-s_1)}{(s_1-s)(s_2-s_a)}\right]^{1/2}\right)\Big/\left(1-i\left[\frac{(s_2-s)(s_a-s_1)}{(s_1-s)(s_2-s_a)}\right]^{1/2}\right)\right\}$$
$$-\frac{2i}{((s_1-s)(s_2-s))^{1/2}}\ln\left\{\left(1-\left[\frac{(s_1-s)(s_2-4)}{(s_2-s)(s_1-4)}\right]^{1/2}\right)\Big/\left(1+\left[\frac{(s_1-s)(s_2-4)}{(s_2-s)(s_1-4)}\right]^{1/2}\right)\right\}-2\pi/[-K(s)]^{1/2}. \quad (13)$$

As W increases further, s_a moves further into the lower half plane, and s_1 moves to the right of s_0. From Eq. (12), we notice that $s=s_1$ is now a singularity of the spectral function, and hence of F on its second sheet (it is a "second-type" singularity.[23]) But from Eq. (13), we see directly that s_0 is not a singularity of F, and further continuation in W is trivial.

We remark that s_a is indeed now a singularity of F, although one can see qualitatively that its effect is likely to be small. If, as is the case, the imaginary part of I^2 is small compared to the real part, s_a will lie close to the real axis; but although it is on the physical sheet, it is on the lower (*un*physical) side of the cut.

V. NUMERICAL RESULTS

The integrals in Eqs. (8) and (13) have been done, using complex arithmic FORTRAN, on the IBM 7094 at BNL. Our purpose is to investigate by explicit calculation the effect of the logarithmic singularities: of s_a on the high mass end of the $\pi\pi$ spectrum, and of s_b on the low mass end, as W varies.

In Fig. 11(a) we have plotted the square of the amplitude, $|F|^2$, versus s, for some typical values of W. (A factor $1/\pi$ is suppressed from now on.) This, and Figs. 11(b), 12(a) and (b), and 14, which we shall describe presently, should be looked at in conjunction with Fig. 6. We see that s_a produces no effect, but that for $W\approx 10.0$ there is a characteristic rise in $|F|^2$ near $s=4$, a result similar to that of Halpern and Watson[6]. From Fig. 6 we see that this W is just the one for which s_b is near the real axis in the lower half plane.

To make the correspondence between the position of s_b and the enhancement of $|F|^2$ still clearer, we have repeated the calculation with an unrealistically small width for I: We took $I=8.91-0.02i$, so that the motion of s_a and s_b is now given by the dashed lines of Fig. 6. For this case, s_b approaches near the real axis from below, at a point $s\approx 5$, distinct from $s=4$; and indeed

[22] See also a recent UCLA preprint by C. Fronsdal and R. E. Norton, 1963 (unpublished), although these authors have not explicitly considered a complex internal mass.

[23] D. B. Fairlie, P. V. Landshoff, J. Nuttall, and J. C. Polkinghorne, J. Math. Phys. 3, 594 (1962).

$|F|^2$ shows—Fig. 11(b)—a definite peak for the appropriate W, near $s=5$.

The effect of s_b, when present, is masked for two reasons: First, even when I_2 is small, s_b approaches the real axis rather near to $s=4$, where the spectral function has to vanish in any case; second, as I_2 increases, s_b moves even nearer to $s=4$ whenever it is just below the real axis, so that this suppression is aggravated. In an actual calculation, therefore, the width may play an important part.

It is instructive to see how the spectral function varies at the interesting values of W. For $W<11.62$, it is just f, calculated from Eq. (10). Figures 12(a) and (b) show, respectively, the real and imaginary parts of f, f_r and f_i. The solid curves are for $I=8.91-0.32i$, the dashed ones for $I=8.91-0.02i$. Referring again to Fig. 6, we see that as W increases from 9.0, for the W such that s_b is just below the real axis, both f_r and f_i rise steeply from zero [see especially the dashed curve in Fig. 12(b) for $W=10.0$, which shows a separate peak near $s=5$], while for larger values of W this becomes less pronounced. For W such that s_a is near the real axis, there is a peak near s_a; nevertheless there is no resultant peak in F.

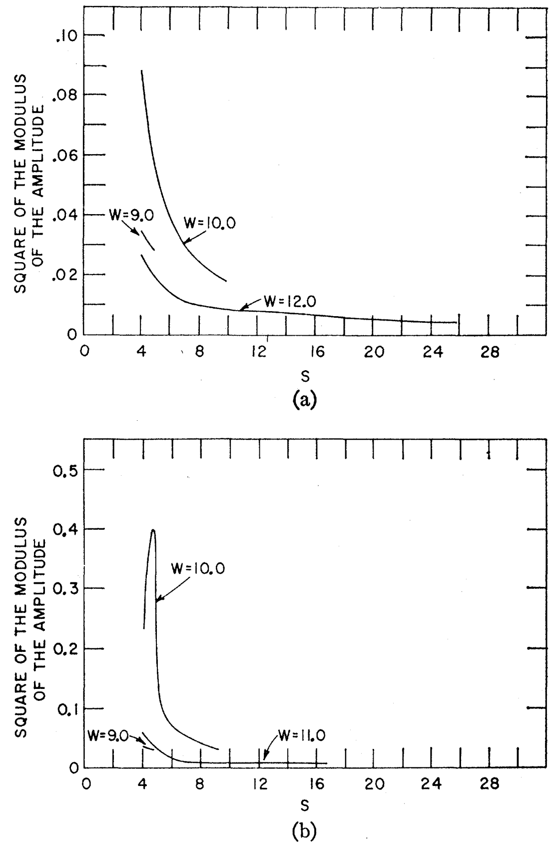


FIG. 11. The square of the amplitude of Fig. 2 versus s for various values of W. (a) for $I=8.91-0.32i$ (b) for $I=8.91-0.02i$.

VI. DISCUSSION

We have calculated, from a dispersion relation in s, the amplitude F of Fig. 2, taking all particles to be spinless isoscalars. The effects of singularities of the weight function were examined, and it was found that an enhancement near $s=4$ was expected, but only for a restricted range in W near the $\pi+I$ threshold; the magnitude of the effect is very sensitive to the width of the unstable particle.

We now wish to consider F as a function of W rather than s— that is, we regard it as a contributor to inelastic πN scattering. We can imagine calculating F from a dispersion relation in W, the contour of which is taken along some path starting at the "two-particle" threshold $W^2=(I+1)^2$, and going to infinity. Such two particle cuts—originating from a state in which one of the particles is unstable—have been considered by Zwanziger,[15] who, as we mentioned in Sec. III, has shown that they are two-sheeted, and reached by a path crossing the inelastic $\pi\pi N$ cut. For the present purposes, we may disregard the fact that I is complex, and treat this πI cut—and the dispersion relation contour—as being along the real axis, just below the physical region, as in the usual case. The spectral function ϕ then involves the s-wave projection of the one-pion-exchange pole in the reaction $\pi I\rightarrow\pi\pi N$ (compare footnote 20), and the question arises as to whether the Peierls mechanism[11] operates to give an enhancement. It is easy to verify that ϕ has exactly the singularities given by Eq. (3); in the W plane we call them W_a and W_b, and they are functions of s. Their motion as s varies may be read off from Fig. 4, and is similar to that of s_a and s_b already described. Namely, for a range in s, W_b lies below the cut, near the real axis, but for no other s do either W_a or W_b lie near the physical region. This range in s is $4\leq s\leq[(I+1)^2-M^2]/I\approx 6$. We can express this condition more generally and concisely in terms of the variables s, y and z of Secs. II and III: a singularity in y is near the physical region if x lies in the range

$$-1\geq x\geq -z, \tag{14}$$

where z is the variable associated with the stable external particle. For the case in which the masses are real, this region of x corresponds to the part of the dashed line in Fig. 4 which is below the real axis.[24] It is clear that for Eq. (14) to be satisfied, z has to be greater than 1, implying that an internal particle is unstable.

This region is easily interpreted on the Dalitz plot picture of the singularities given in Sec. II. Figure 13 is a Dalitz plot for $\sqrt{s}$ and $\sqrt{t}$, where $\sqrt{t}$ is the mass of the intermediate πN state. Suppose there are resonances at $\sqrt{s}=\rho$, the ρ meson, and $\sqrt{t}=I$, the (3,3) isobar. For small $W=W_0$, the two resonance bands $\sqrt{s}=\rho$, $\sqrt{t}=I$ (shown as double lines in Fig. 13) do not intersect in the allowed physical region R: The resonances cannot be

[24] The special nature of this region was first mentioned by G. Barton and C. Kacser, Nuovo Cimento 21, 593 (1961); see also Ref. 6.

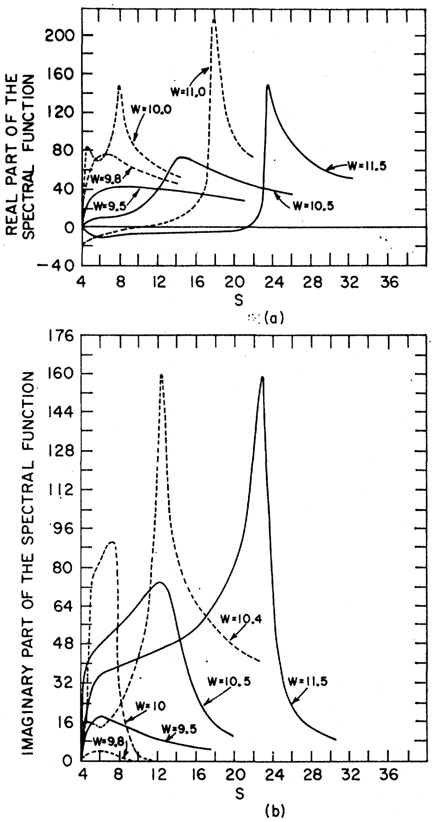
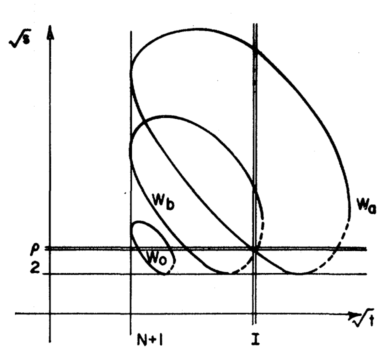


FIG. 12. The real (a) and the imaginary (b) parts of the spectral function in units of μ^{-2} versus s in units of μ^2 (μ=pion mass). The solid curve is for $I=8.91-0.32i$, the dashed for $I=8.91-0.02i$.

simultaneously produced for that W. As W increases, the point of crossing will appear on the edge of R for $W=W_b$, and then drop off R at $W=W_a$. As the names imply, W_a and W_b are the positions of the logarithmic singularities of F as a function of W, so we may call these "double excitation" thresholds. The content of Eq. (14) is then that the higher such threshold gives no effect, while the lower does so only if the bands cross on the boundary R in the lower right-hand segment α shown dotted in Fig. 13.

To illustrate this effect in W, we fix s at a value of 4.1, and calculate F for a range of values of W, from Eqs. (8) and (10). Figure 14 shows $|F|^2$ versus W; the

FIG. 13. Dalitz plots in the $\sqrt{s}$, $\sqrt{t}$ plane for various values of W.



solid curve is, as before, for $I=8.91-0.32i$ and the dashed one is for $I=8.91-0.02i$. For the narrow width it is especially clear: Firstly, there is a peak at $W\approx 10.4$, which is just the value for which $s=4.1$ gives a singularity in W near the physical region [the $\sqrt{s}=(4.1)^{1/2}$, $\sqrt{t}=I$ bands crossing on α]; secondly, one sees a separate shoulder at $W\approx 10$, corresponding to the normal threshold $I+1$. The latter is just the cusp phenomenon of Nauenberg and Pais,[12] while the former is an additional enhancement from W_b. Unfortunately, no distinction between the phenomena remains for the realistic width, and the effect is much reduced.[25] It disappears

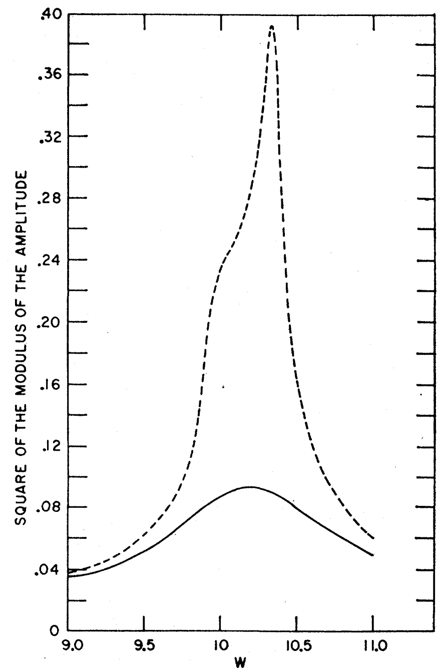


FIG. 14. The square of the amplitude versus W, for $s=4.1$; the solid curve is for $I=8.91-0.32i$, the dashed is for $I=8.91-0.02i$. The πI threshold is at $W\approx 10$.

altogether in both cases as soon as s increased beyond $s=5$, the bands no longer crossing on α. In summary, the effect, if observable, should show up as a bump in the production process, if the two pions are observed near threshold. (See also Ref. 2.)

Our conclusion is, therefore, that it is only for a limited range of the variables, Eq. (14), that a double excitation threshold of the triangle graph gives any effect. This criterion is quite general, and we may apply it to the graph shown in Fig. 15, for example. This process appears to be closely related to that originally considered by Peierls,[11] in that the structure in the W channel comes from the nucleon exchange pole in the reaction $\pi I\rightarrow\pi I$. We find, however, that Eq. (14) is

[25] We stress that the actual calculation was not a dispersion relation in W, but one in s, Eq. (8), evaluated for one value of s and several values of W; hence it does not depend on assumptions regarding analyticity in the W plane.

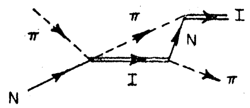


FIG. 15. An approximation for the production amplitude $\pi N \to \pi I$.

not satisfied, as has already been pointed out by Goebel,[26] so that no enhancement of the inelastic amplitude is expected from this graph.

We ask, finally, what effects may be found in the elastic channel, $\pi N \to \pi N$. Consider again the πI state as a contributor to the absorptive part of the $\pi N \to \pi N$ reaction (Fig. 16), as may be a reasonable approxima-

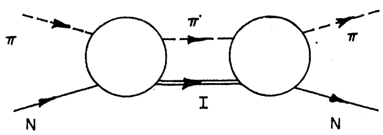


FIG. 16. The πI contribution to the absorptive part of the elastic process.

tion near the πI threshold. This contribution contains the production vertex $\pi N \to \pi I$, which itself may be thought of as proceeding via the πI intermediate state, in that energy range, so that it is given by Fig. 14. The result is then Fig. 17. Now, we expect the elastic

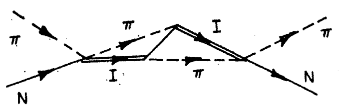


FIG. 17. The approximation of Fig. 15 inserted into Fig. 16.

amplitude to be enhanced when the production vertex is, but we have already seen that, for Fig. 15, it is not. It appears unlikely that this single graph can be responsible for any effect in the elastic channel. This conclusion is essentially the same as that reached by different methods by Hwa.[27] This is not to say, however, that repeated iterations of the singularity, through unitarity relations of the type considered by Hwa in Secs. VI and VII of his paper, could not lead to the formation of a suitable resonance pole.[28] This possibility is perhaps suggested by the work of Peierls and Tarski,[8] in which the complete solution of a model seems to show some double excitation effects.

Although Eq. (14) is not satisfied for the case just discussed, it is not hard to find examples for which it is. Consider, for instance, the process of Fig. 18, in which

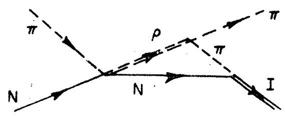


FIG. 18. A graph leading to enhancement of the elastic channel $\pi N \to \pi I$.

a ρ is produced, decays, and a πI state is formed. If z and x are associated with the external pion and isobar, respectively, it is readily verified that Eq. (14) holds. The resulting enhancement in W occurs near the $N\rho$ threshold. It is at least possible that this may account for the 1688 πN resonance which, as is well known, lies close to this threshold.[29] This mechanism is then a kind of synthesis of the Ball-Frazer and Peierls suggestions. It has been observed[30] that the levels of the known nucleon isobars appear to be separated by the masses of certain combinations of pion resonance states. It is certainly possible to construct graphs such as Fig. 18, satisfying Eq. (14), all giving enhancements near the appropriate (nucleon resonance+pion resonance) threshold. To decide if this observation provides a basis for understanding the empirical level spacing or not would require a much more elaborate calculation, in which, at least, the essential complications of spin were properly treated.

ACKNOWLEDGMENTS

It is a pleasure to thank my colleagues Dr. T. L. Trueman and Dr. T. Yao, who participated in the early stages of this work, for many interesting discussions on this and many other topics. I am also indebted to Professor S. F. Tuan for enthusiastically encouraging the present investigation.

[26] C. Goebel, University of Wisconsin preprint, 1962 (unpublished).
[27] R. C. Hwa, Phys. Rev. **130**, 2580 (1963).
[28] R. F. Peierls (private communication).
[29] This correlation has been emphasized by S. F. Tuan, Nuovo Cimento **23**, 448 (1962); see also R. M. Sternheimer, Phys. Rev. Letters **10**, 309 (1963), for this and other threshold coincidences.
[30] T. F. Kycia and K. F. Riley, Phys. Rev. Letters **10**, 266 (1963).